Some Results from Microeconomics

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I. Consumer

1. Utility maximization

Utility function: $u = u(x_1, x_2)$ $\quad$ $u : \mathbb{R}^2_+ \to \mathbb{R}$

(i) twice continuously differentiable.

(ii) $u_1 \equiv \dfrac{\partial u}{\partial x_1} > 0$ and $u_2 \equiv \dfrac{\partial u}{\partial x_2} > 0$

(iii) strictly quasi-concave.

♣ Concavity and quasi-concavity

1. Concave functions

Let X be a convex subset of $\mathbb{R}^n$. Namely, for all $\boldsymbol{x}, \boldsymbol{y} \in X$ and $\alpha \in [0, 1]$

$\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y} \in X$

(Def)(Concave function) $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ is **concave** if for all $\boldsymbol{x}, \boldsymbol{y} \in X$ and $\alpha \in [0, 1]$

$f(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \geq \alpha f(\boldsymbol{x}) + (1-\alpha) f(\boldsymbol{y})$

(Thm)(Properties of concave functions)

[Note: f is called n-times continuously differentiable if $f^{(n)}$ is continuos.]

(i) If f is concave, f is continuos.

(ii) If f and g are concave, then $f+g$ is concave.

(iii) If $f : X \subseteq \mathbb{R} \to \mathbb{R}$ is continuously differentiable, then f is concave iff

$f(y) - f(x) \leq f'(x)(y-x)$ for all $x, y \in X$

More generally, if $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable, then f is concave iff $f(\boldsymbol{y}) - f(\boldsymbol{x}) \leq Df(\boldsymbol{x})(\boldsymbol{y} - \boldsymbol{x})$ for all $\boldsymbol{x}, \boldsymbol{y} \in X$

[Note: $Df(\boldsymbol{x}) = \left(\dfrac{\partial f}{\partial x_1}, \dfrac{\partial f}{\partial x_2}, \cdots \dfrac{\partial f}{\partial x_n}\right)$, the Jacobian of f at $\boldsymbol{x}$.]

(Corol) Let $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ be concave and continuously differentiable. If $Df(\boldsymbol{x_o}) = \boldsymbol{0}$ at $\boldsymbol{x_o} \in X$, then f achieves its global maximum at $\boldsymbol{x_o}$.

(iv) If $f : X \subseteq \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable, then f is concave iff $f''(x) \leq 0$ for all $x \in X$.

More generally, if $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ is twice continuously differentiable, then

f is concave iff $D^2 f$ is a negative semidefinite matrix.

2. Quasi-concave functions

(Def)(Quasi-concave function) A function $f : \mathbb{R}^n \to \mathbb{R}$ is quasi-concave if

$f(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \geq \min\{f(\boldsymbol{x}), f(\boldsymbol{y})\}$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and $\alpha \in [0, 1]$

(Thm) $f : \mathbb{R}^n \to \mathbb{R}$ is quasi-concave iff the upper contour set

$\{\boldsymbol{x} \in \mathbb{R}^n \mid f(\boldsymbol{x}) \geq a\}$ is convex for all $a \in \mathbb{R}$.

(Thm) $f : \mathbb{R}^n \to \mathbb{R}$ is quasi-concave iff $Df(\boldsymbol{x})(\boldsymbol{y} - \boldsymbol{x}) \geq 0$ whenever $f(\boldsymbol{y}) \geq f(\boldsymbol{x})$

(Thm) If f is concave then f is quasi-concave.

[But the converse is not necessarily true.]

(True or false?)

(i) If f is concave, then its upper contour set is convex.

(ii) $f(x) = e^x$ is concave.

(iii) $f(x) = \log(x)$ is concave.

(iv) $f(x) = x^3$ is quasi-concave.

(v) Concave utility functions satisfy the law of diminishing marginal utilities.

(vi) Quasi-concave utility functions do not necessarily satisfy the law of diminishing marginal utilities.

Utility maximization problem

$$\underset{x_1, x_2}{Max} \quad u(x_1, x_2)$$

$$\text{s.t.} \qquad p_1 x_1 + p_2 x_2 = I$$

<Method 1> Intuitive (or graphical) approach

Indifference curve (for any given level u_o of utility): $u(x_1, x_2) = u_o$

Slope of IC:

[By taking total derivative of the IC and rearranging the result]

$$\left.\frac{dx_2}{dx_1}\right|_{u = u_o} = -\frac{u_1}{u_2} \equiv -MRS$$

Slope of the BC

$$\frac{dx_2}{dx_1} = -\frac{p_1}{p_2}$$

Hence the necessary conditions are:

(i) Two slops must coincide: $u_1 = \frac{p_1}{p_2} u_2$

(ii) BC must be satisfied: $p_1 x_1 + p_2 x_2 = I$

<Method 2> Plug-in Method

Rewrite the BC: $x_2 = \frac{I}{p_2} - \frac{p_1}{p_2} x_1$

and plug into the UF to obtain a unconstrained maximization problem:

$$\underset{x_1}{Max} \ u(x_1, \ \frac{I}{p_2} - \frac{p_1}{p_2} x_1)$$

FOC

$$\frac{d}{dx_1} u(x_1, \ \frac{I}{p_2} - \frac{p_1}{p_2} x_1) = u_1 + u_2 \left(- \frac{p_1}{p_2} \right) = 0$$

$\Rightarrow$ (i) $u_1 = \frac{p_1}{p_2} u_2$

and (ii) $x_2 = \frac{I}{p_2} - \frac{p_1}{p_2} x_1$

<Method 3> Lagrangian (Most useful when there are more variables.)

$$\mathcal{L} = u(x_1, x_2) + \lambda(I - p_1 x_1 - p_2 x_2)$$

FOC $u_1 - \lambda p_1 = 0$

$u_2 - \lambda p_2 = 0$

$I - p_1 x_1 - p_2 x_2 = 0$

Eliminating λ by combining the first two equations gives the same necessary conditions as before:

$$u_1 = \frac{p_1}{p_2} u_2$$

$$p_1 x_1 + p_2 x_2 = I$$

Solving the FOC then gives **(Marshallian) Demand functions**

$$x_1^m = x_1^m(p_1, p_2, I)$$

$$x_2^m = x_2^m(p_1, p_2, I)$$

(Exercise)

(i) Show that the Marshallian demand functions are homogenous of degree zero. (This is really obvious from the structure of the utility maximization problem.)

(ii) Show that any positive monotonic transformation of the utility function will not affect the demand functions. ■

(Note) The graph of $x_1^m = x_1^m(p_1, p_2, I)$ on x_1-p_1 plane given p_2 and I is called the individual **demand curve** of x_1. Market demand curve is the horizontal aggregation of the individual demand curves.

(Example) Cobb-Douglas utility function

$$u(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$$

Marginal utilities:

$$u_1 = \alpha\left(\frac{x_2}{x_1}\right)^{1-\alpha}, \quad u_2 = (1-\alpha)\left(\frac{x_1}{x_2}\right)^{\alpha}$$

Lagrangian $\quad \mathcal{L} = x_1^{\alpha} x_2^{1-\alpha} + \lambda(I - p_1 x_1 - p_2 x_2)$

FOC
$$\frac{\partial \mathcal{L}}{\partial x_1} = \alpha\left(\frac{x_2}{x_1}\right)^{1-\alpha} - \lambda p_1 = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = (1-\alpha)\left(\frac{x_1}{x_2}\right)^{\alpha} - \lambda p_2 = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = I - p_1 x_1 - p_2 x_2 = 0$$

FOC w/o λ:
$$\alpha\left(\frac{x_2}{x_1}\right)^{1-\alpha} = \frac{p_1}{p_2}(1-\alpha)\left(\frac{x_1}{x_2}\right)^{\alpha}$$

$$I - p_1 x_1 - p_2 x_2 = 0$$

Demand functions
$$x_1^m = \alpha\frac{I}{p_1}$$

$$x_2^m = (1-\alpha)\frac{I}{p_2}$$

(Exercise) $x_1^m = \alpha\frac{I}{p_1}$ above is the individual demand function. Assuming that all consumers have the same Cobb-Douglas utility function, obtain the market demand function. ■

The envelope theorem

Let's consider a simple problem that we want to maximize $f(x, c)$ which is $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ Here x is the choice variable (and thus endogenous) whereas c is exogenous.

Let $v(c)$ be the maximized value of f and x^* be the value of x at which f attains the maximum. Hence,

$$\frac{\partial f}{\partial x} = 0 \text{ at } x = x^* \text{ and}$$

$$v(c) = f(x^*(c), c)$$

Now, we ask how much $v(c)$ will be affected due to a change in c:

$$v'(c) = \frac{\partial f}{\partial x}\frac{dx^*}{dc} + \frac{\partial f}{\partial c}$$

(In the right hand side, we have two terms which are respectively 'indirect' and 'direct' effects.) where x is evaluated at x^* so that $\frac{\partial f}{\partial x} = 0$ and the first term vanishes. Hence

$$v'(c) = \frac{\partial f}{\partial c} \quad \text{which in full expression is } v'(c) = \frac{\partial}{\partial c} f(x^*(c), c)$$

Namely, we only have the direct effect. This fact is referred to as the **envelope theorem.**

(Note on notation) In this note, by starred variables such as x^* and λ^* we mean those obtained as solutions to some optimization problem. When the variables have special names such as Marshallian demand, we may use a special notation such as x^m. So, from the point of view of pure maximization, x^m may be written as x^*.

(Example) An application of the envelope theorem

The Lagrangian of the utility maximization problem:

$$\mathcal{L} = u(x_1, x_2) + \lambda(I - p_1x_1 - p_2x_2)$$

Consider now the Lagrangian at the optimal value of endogenous variables x_1^*, x_2^*, and λ^*. The optimal value is the function of exogenous variables, so we may write

$$\mathcal{L}^*(p_1, p_2, I) = u(x_1^*, x_2^*) + \lambda^*(I - p_1x_1^* - p_2x_2^*)$$

We then ask how much $\mathcal{L}^*$ will be affected when there is a change in I. Although we know that x_1^*, x_2^*, and λ^* all depend on I, the envelope theorem tells us that only the direct effect matters. Hence

$$\frac{\partial}{\partial I}\mathcal{L}^*(p_1, p_2, I) = \lambda^*$$

Notice that at the optimum, the budget constraint holds and so $\mathcal{L}^* = u(x_1^*, x_2^*)$. Hence λ^* can be interpreted as the marginal utility of income (or, marginal utility of money as often said in microeconomics). ■

(Exercise)

(i) (Answer true or false, and then explain) According to the envelope theorem, $\frac{\partial}{\partial I}u(x_1^*(p_1, p_2, I), x_2^*(p_1, p_2, I)) = 0$ because there is no direct effect of I on u.

(ii) Consider the plug-in method and let the maximum utility be $u^* = u(x_1^*, \frac{I}{p_2} - \frac{p_1}{p_2}x_1^*)$. Find $\frac{\partial u^*}{\partial I}$ and then using the first order conditions in the Lagrangian method $u_j - \lambda p_j = 0\,(j = 1, 2\,)$, show that $\frac{\partial u^*}{\partial I} = \lambda^*$.

■

2. Comparative statics

Indirect utility function

$$v(p_1, p_2, I) \;=\; u(x_1^m(p_1, p_2, I),\; x_2^m(p_1, p_2, I))$$

which is the maximum utility achievable under the given p_1, p_2, I.

Expenditure minimization

Let e be the expenditure on x_1 and x_2 so that $e = p_1x_1 + p_2x_2$. Consider then the expenditure minimization problem given a target utility level $\overline{u}$.

$$Min_{x_1, x_2}\;\; p_1x_1 + p_2x_2$$
$$\text{s.t. } u(x_1, x_2) = \overline{u}$$

[Endogenous variables = x_1, x_2 Exogenous variables = p_1, p_2, $\overline{u}$]

Lagrangian: $\mathcal{L} = p_1x_1 + p_2x_2 + \theta(\overline{u} - u(x_1, x_2))$

FOC

$$p_1 - \theta u_1 = 0$$
$$p_2 - \theta u_2 = 0$$
$$\overline{u} - u(x_1, x_2) = 0$$

FOC w/o θ:

$$u_1 = \frac{p_1}{p_2}u_2$$
$$\overline{u} - u(x_1, x_2) = 0$$

Solving the FOC yields the Hicksian demand function

$$x_1^h = x_1^h(p_1, p_2, \bar{u})$$
$$x_2^h = x_2^h(p_1, p_2, \bar{u})$$

(Note: Due to the presence of unobservable $\bar{u}$, Hicksian demands are not observable.)

Notice that the condition $u_1 = \frac{p_1}{p_2} u_2$ is common in both utility maximization(UM) and expenditure minimization(EM) problems. Hence if either $\bar{u}$ is chosen to be the same as the maximum utility level in UM, or e(expenditure) is chosen to be the same as I, then both problems give exactly the same solutions x_1^* and x_2^*. More precisely, we have

$$v(p_1, p_2, e(p_{1,} p_{2,} \bar{u})) = \bar{u}$$
$$e(p_1, p_2, v(p_1, p_2, I)) = I$$

and, for $j = 1, 2$,

$$x_j^m(p_1, p_2, I) = x_j^h(p_1, p_2, v(p_1, p_2, I)) = x_j^*$$
$$x_j^h(p_1, p_2, \bar{u}) = x_j^m(p_1, p_2, e(p_1, p_2, \bar{u})) = x_j^*.$$

Now we take the partial derivative of the last equation with respect to p_k in order to obtain analytically the relation

(price effect(PE)) = (substitution effect(SE)) + (income effect(IE)).

$$\frac{\partial x_j^h}{\partial p_k} = \frac{\partial x_j^m}{\partial p_k} + \frac{\partial x_j^m}{\partial I}\frac{\partial e}{\partial p_k} \qquad \text{for } j = 1,2 \text{ and } k = 1,2$$

[because by the envelope theorem $\frac{\partial e}{\partial p_k} = x_k^*$]

$$= \frac{\partial x_j^m}{\partial p_k} + x_k^* \frac{\partial x_j^m}{\partial I}$$

So, we get the following that is known as the **Slutsky equation.**

$$\frac{\partial x_j^m}{\partial p_k} = \frac{\partial x_j^h}{\partial p_k} - x_k^* \frac{\partial x_j^m}{\partial I} \qquad \text{for } j = 1,2 \text{ and } k = 1,2$$

which shows the relation '(PE)=(SE)+(IE)'.

Classification of goods

(i) Normal good: $\frac{\partial x^m}{\partial I} \geq 0$ [Inferior good: $\frac{\partial x^m}{\partial I} < 0$]

(ii) Gross substitutes: $\frac{\partial x_i^m}{\partial p_j} > 0$

(iii) Giffen good: $\dfrac{\partial x_j^m}{\partial p_j} > 0$

[The Slutsky equation says $\dfrac{\partial x_j^m}{\partial p_j} = \dfrac{\partial x_j^h}{\partial p_j} - x_j^* \dfrac{\partial x_j^m}{\partial I}$ and we know $\dfrac{\partial x_j^h}{\partial p_j} < 0$. Hence for x_j to be a Giffen good, the income effect $(-x_j^* \dfrac{\partial x_j^m}{\partial I})$ must be large and positive to offset the SE.]

(iv) Luxury good is a good whose income elasticity is greater than one:

$$\left(\frac{dx^m}{x^m}\right)/\left(\frac{dI}{I}\right) = \left(\frac{dx^m}{dI}\right)/\left(\frac{x^m}{I}\right) > 1$$

(An equivalent definition) Luxury good is a good whose budget share increases as income grows.

II. Pareto Efficiency and Competitive Equilibrium

(Thm)(First welfare theorem) Every competitive allocation is Pareto efficient.

(Thm)(Second welfare theorem) Every Pareto efficient allocation can be decentralized as a competitive equilibrium.